Analogues of Cliques for (m,n)-colored Mixed Graphs

Julien Bensmail, Christopher Duffy, Sagnik Sen

Analogues of Cliques for (m, n)-colored Mixed Graphs

Julien Bensmail[a], Christopher Duffy[b], Sagnik Sen[c]

(a) INRIA and Université Nice-Sophia-Antipolis, I3S, UMR 7271, 06900 Sophia-Antipolis, France
(b) Dalhousie University, Halifax, Canada
(c) Indian Institute of Science, Bangalore, India

Abstract

An (m, n)-colored mixed graph is a mixed graph with arcs assigned one of m different colors and edges one of n different colors. A homomorphism of an (m, n)-colored mixed graph G to an (m, n)-colored mixed graph H is a vertex mapping such that if uv is an arc (edge) of color c in G, then $f(u)f(v)$ is also an arc (edge) of color c. The (m, n)-colored mixed chromatic number, denoted $\chi_{m,n}(G)$, of an (m, n)-colored mixed graph G is the order of a smallest homomorphic image of G. An (m, n)-clique is an (m, n)-colored mixed graph C with $\chi_{m,n}(C) = |V(C)|$. Here we study the structure of (m, n)-cliques. We show that almost all (m, n)-colored mixed graphs are (m, n)-cliques, prove bounds for the order of a largest outerplanar and planar (m, n)-clique and resolve an open question concerning the computational complexity of a decision problem related to $(0, 2)$-cliques. Additionally, we explore the relationship between $\chi_{1,0}$ and $\chi_{0,2}$.

Keywords: colored mixed graphs, signed graphs, graph homomorphisms, chromatic number, clique number, planar graphs

1 Introduction

Coloring Mixed Graphs

The notions of vertex coloring and chromatic number were generalized by Nešetřil and Raspaud [11] by defining (m, n)-colored mixed graphs and colored graph homomorphisms. This notion of homomorphism captures the definition of homomorphism for graphs, oriented graphs and edge-colored graphs.

A *mixed graph* is a simple graph in which a subset of the edges have been oriented to be arcs. An *(m, n)-colored mixed graph*, $G = (V, A \cup E)$, with vertex set V, arc set A, and edge set E, is a mixed graph where each $uv \in A(G)$ is colored with one of m colors $\{1, 2, 3, \dots, m\}$ and each $wx \in E(G)$ is colored with one of n colors $\{1, 2, 3, \dots, n\}$. When $m = 0$ (resp. $n = 0$) it is assumed that the mixed graph used to form the $(0, n)$-colored mixed graph (resp. $(m, 0)$-colored mixed graph) contains no arcs (resp. edges). From this we see that a $(0, 1)$-colored mixed graph is a simple graph, a $(0, k)$-colored mixed graph is a k-edge-colored graph and a $(1, 0)$-colored mixed graph is an oriented graph.

As an (m,n)-colored mixed graph is a mixed graph with decorated edges, we observe that each (m,n)-colored mixed graph G has an underlying simple graph, which we denote by $U(G)$. If $uv \in E(U(G))$, then the *adjacency type* of uv is an *edge colored* i if $uv \in E$ and uv has color i, or an *arc colored* j if $uv \in A$ and uv has color j.

In discussing arcs and edges of (m,n)-colored mixed graphs we make no distinction in notation between arcs and edges. Since each pair of adjacent vertices in $U(G)$ has at most one adjacency type in G, there is no possibility for confusion in the notation uv being used to refer to either an arc from u to v or an edge between u and v, as the case may be. We say that $uv, wx \in A \cup E$ have the same adjacency type if

- $uv, wx \in A$ and both have color $i \in \{1, 2, 3, \ldots, m\}$;
- $vu, xw \in A$ and both have color $i \in \{1, 2, 3, \ldots, m\}$; or
- $uv, wx \in E$ and both have color $j \in \{1, 2, 3, \ldots, n\}$.

Let G and H be (m,n)-colored mixed graphs. A *colored homomorphism* of G to H is a function $f : V(G) \to V(H)$ such that the adjacency type of uv in G is the same as that of $f(u)f(v)$ in H, for all $uv \in E(U(G))$. That is, a colored homomorphism is a vertex mapping that preserves colored edges and colored arcs [11]. We write $f : G \to H$ when there exists a homomorphism, f, of G to H, or $G \to H$ when the name of the function is not important. Finally, we say that H is a *homomorphic image* of G.

The (m,n)*-colored mixed chromatic number* of G, denoted $\chi_{m,n}(G)$, is the least integer k such that there exists a homomorphic image of G of order k. For a simple graph Γ, we let $\chi_{m,n}(\Gamma)$ denote the maximum (m,n)-colored mixed chromatic number over all (m,n)-colored mixed graphs G such that $U(G) = \Gamma$. For a family of undirected simple graphs $\mathcal{F}$, we let $\chi_{m,n}(\mathcal{F})$ denote the maximum of $\chi_{m,n}(\Gamma)$ taken over all $\Gamma \in \mathcal{F}$.

We note that letting $m = 0$ and $n = 1$ in the definitions above gives the usual definitions of graph homomorphism and chromatic number. Similarly, letting $m = 1$ and $n = 0$ gives the definitions of oriented graph homomorphism and oriented chromatic number considered by many researchers over the last two decades. We refer the reader to [14] for a survey of results in this area. Further, taking $m = 0$ and $n = k$ gives the definition of homomorphism used by many authors in the study of homomorphisms of k-edge-colored graphs [1, **?**, 8].

Cliques for Mixed Graphs

An (m,n)*-clique* C is an (m,n)-colored mixed graph for which $\chi_{m,n}(C) = |V(C)|$. The (m,n)*-absolute clique number* of an (m,n)-colored mixed graph G, denoted $\omega_{a(m,n)}(G)$, is the largest k such that G contains an (m,n)-clique of order k. As above, we note that when $m = 0$ and $n = 1$, the definitions above give exactly those for clique and clique number; and when $m = 1$ and $n = 0$, the definitions above give exactly those for oriented clique [5] and oriented absolute clique number [9]. In Section 2 we show that (m,n)-cliques are not rare objects. In fact, for $(m,n) \neq (0,1)$ nearly every (m,n)-colored mixed graph is an (m,n)-clique.

In previous studies of oriented cliques [9, 13], a related parameter, the oriented relative clique number, arose as a useful tool in studying the oriented chromatic number. Here we provide a generalization of this parameter for (m,n)-colored mixed graphs. A subset $R \subseteq V(G)$ is a *relative* (m,n)*-clique* of an (m,n)-colored mixed graph G if for every pair of distinct vertices

$u, v \in R$ and every homomorphism $f : G \to H$, we have $f(u) \neq f(v)$. That is, no two distinct vertices of a relative clique can be identified under any homomorphism. The (m, n)*-relative clique number*, denoted $\omega_{r(m,n)}(G)$, of an (m, n)-colored mixed graph G is the cardinality of a largest relative (m, n)-clique of G. From the definitions it is clear that

$$\omega_{a(m,n)}(G) \leq \omega_{r(m,n)}(G) \leq \chi_{m,n}(G).$$

For undirected simple graphs, the (absolute) clique number and the relative clique number coincide; however, this is not the case in when $(n, m) \neq (0, 1)$ [13]. In Section 3 we study $\omega_{a(m,n)}$ and $\omega_{r(m,n)}$ for the families of planar and outerplanar (m, n)-colored mixed graphs and make a conjecture regarding the order of the largest planar (m, n)-clique.

The computational complexity of graph homomorphism problems has been examined by a variety of authors in many different contexts [1, **?**, **?**, 6]. For the family of simple undirected graphs, a dichotomy theorem exists for complexity of the H-coloring problem (see [**?**]). Recent work in both oriented graphs and k-edge-colored graphs suggests that such a dichotomy theorem may exist for the H-coloring problem for these families of graphs [**?**, **?**, **?**].

Naserasr, Rollovà and Sopena [10] recently studied homomorphisms between equivalence classes of $(0, 2)$-graphs called *signed graphs*. In their study they reformulated and extended several classical theorems and conjectures of graph theory including the Four-Color Theorem and Hadwiger's conjecture. A $(0, 2)$-colored mixed graph is a signed clique if each pair of vertices is either adjacent or is part of a 4-cycle with three edges of the same color while the other edge has a different color. In [**?**] Naserasr asks the following question:

Question 1.1. *Given an undirected simple graph, what is the complexity of deciding if it is the underlying graph of a signed clique?*

Given an undirected simple graph, deciding if it is the underlying graph of an (m, n)-clique for $(m, n) = (1, 0)$ and $(0, 2)$ is known to be NP-complete [2]. In Section 4 we fully answer Question 1.1 by showing that for a given simple undirected graph Γ, deciding if there exists a signed clique G such that $U(G) = \Gamma$ belongs to the class of NP-complete decision problems.

In their work introducing the (m, n)-colored mixed chromatic number, Nešetřil and Raspaud showed that if G has acyclic chromatic number at most t, then $\chi_{m,n}(G) \leq t(2m+n)^{t-1}$ [11]. This result generalized a similar result for oriented graphs [12] and one for k-edge-colored graphs [1]. As each planar graph admits an acyclic coloring using no more than 5 colors [3], the same result implies $\chi_{m,n}(\mathcal{P}) \leq 5(2m+n)^4$ for the family $\mathcal{P}$ of planar graphs. This, in turn, yields $\chi_{1,0}(\mathcal{P}) \leq 80$ and $\chi_{0,2}(\mathcal{P}) \leq 80$, bounds which had appeared previously in [12] and [1] respectively. That similar techniques and results appear in parallel for oriented graphs and 2-edge-colored graphs suggests the possible existence of a direct relationship between homomorphisms of oriented graphs and homomorphisms of 2-edge-colored graphs (see [**?**] and [**?**], and [12] and [1], for example). In addition to our work on cliques, in this article we explore this speculation and show in fact that there exist simple graphs Γ for which the values of $\chi_{0,2}(\Gamma)$ and $\chi_{1,0}(\Gamma)$ are arbitrarily different (see Section 5). We posit that the appearance of a relationship between homomorphisms of oriented graphs and those of 2-edge-colored graphs comes by way of the unifying theory of colored homomorphism.

The remainder of this paper is structured as follows. In Section 2 we show that almost all (m, n)-colored mixed graphs are (m, n)-cliques. In Section 3 we discuss the (m, n)-relative clique number and the (m, n)-absolute clique number for the families of planar and outerplanar (m, n)-colored mixed graphs. In Section 4 we examine the computational complexity of deciding whether a given undirected graph is the underlying simple graph of some signed clique. Finally, in Section 5 we show that there exist underlying graphs for which the values of the $(0, 2)$-mixed chromatic number and the $(1, 0)$-mixed chromatic number are arbitrarily different.

2 The Structure of (m, n)-cliques

We begin by characterizing (m, n)-cliques. Let G be an (m, n)-colored mixed graph. Let uvw be a 2-path in $U(G)$. We say that uvw is a *special 2-path* if one of the following holds:

(i) uv and vw are edges of different colors,

(ii) uv and vw are arcs (possibly of the same color),

(iii) uv and wv are arcs of different colors,

(iv) vu and vw are arcs of different colors,

(v) exactly one of uv and vw is an edge.

That is, uvw is a *special 2-path* if uv and wv do not have the same adjacency type.

Proposition 2.1. *Let G be an (m, n)-colored mixed graph. A pair of vertices $u, v \in V(G)$ are part of a relative clique if and only if they are adjacent or are joined by a special 2-path.*

Proof. $\Rightarrow$ Let u, v be two vertices of an (m, n)-colored mixed graph G. If they are not part of any relative clique, then there exists an (m, n)-colored mixed graph H and a homomorphism $f : G \to H$ such that $f(u) = f(v)$. If u and v are adjacent, then the image of uv, $f(u)f(v)$, is a loop in H, a contradiction as $U(H)$ is simple. Suppose now that u, v are the ends of a special 2-path uxv. Since $f(u) = f(v)$ and $U(H)$ is simple it must be that ux and vx have the same adjacency type. This contradicts that uxv is a special 2-path. Hence, u, v can neither be adjacent, nor be joined by a special 2-path.

$\Leftarrow$ Assume that u and v are neither adjacent, nor joined by a special 2-path. By identifying u and v and deleting duplicate edges/arcs of the same color, we arrive at an (m, n)-colored mixed graph H. Let x be the vertex formed by identifying u and v. The vertex mapping $g : V(G) \to V(H)$ given by

$$g(z) = \begin{cases} x & \text{if } z = u, v \\ z & \text{otherwise} \end{cases}$$

is a homomorphism of G to H. □

Corollary 2.2. *An (m, n)-colored mixed graph G is an (m, n)-clique if and only if each pair of non-adjacent vertices of G are joined by special 2-path.*

Proof. It follows from the definitions that if (m, n)-colored mixed graph G is an (m, n)-clique if and only if all of its vertices are part of the same relative clique. The result now follows by Proposition 2.1. □

Consider the following model of generating a random (m, n)-colored mixed graph on k vertices. For every pair of vertices u, v there are $1 + 2m + n$ possibilities for adjacency: 1) non-adjacent, 2) an arc, either uv or vu, of one of m possible colors, or 3) an edge of one of n possible colors. A random (m, n)-colored mixed graph on k vertices is generated by selecting, with uniform

probability, one of these $1+2n+m$ possibilities for each pair of vertices. We show that under such a model, as $k \to \infty$ the probability of generating an (m,n)-clique approaches 1.
Theorem 2.3. *For $(m,n) \neq (0,1)$ almost every (m,n)-mixed graph is an (m,n)-clique.*

Proof. Let $\mathcal{G}_k$ be the set of all (m,n)-colored mixed graphs on k vertices (where $(m,n) \neq (0,1)$), and let $\mathcal{C}_k$ be the set of all (m,n)-cliques on k vertices. We show asymptotically almost surely that $G_k \in \mathcal{G}_k$ is an (m,n)-clique. This implies directly that

$$\lim_{k\to\infty} \frac{|\mathcal{C}_k|}{|\mathcal{G}_k|} = 1.$$

For $G_k \in \mathcal{G}_k$ and $u, v \in G_k$, let $X_{u,v}$ be the random variable that is 1 if u and v are not adjacent and also are not the ends of a special 2-dipath and 0 otherwise. If u and v are not the ends of a special 2-dipath and not adjacent, then for each $x \notin \{u,v\}$ there are $6m+3n+1$ possibilities for graph induced by u, x, v.

Observe that

$$\mathbb{E}(X_{u,v}) = Pr(X_{u,v} = 1) < \left(\frac{6m+3n+1}{(2m+n+1)^2}\right)^{k-2}.$$

Let $X = \sum_{u,v\in V(G)} X_{u,v}$. By linearity of expectation we have

$$\mathbb{E}(X) < \binom{k}{2}\left(\frac{6m+3n+1}{(2m+n+1)^2}\right)^{k-2}.$$

Observe that $\mathbb{E}(X) \to 0$ as $k \to \infty$ for fixed $(m,n) \neq (0,1)$. By *Markov's inequality* we have

$$Pr(X \geq 1) \leq \mathbb{E}(X).$$

Thus

$$Pr(X \geq 1) \to 0 \text{ as } k \to \infty.$$

Therefore a.a.s., we have $X = 0$. That is, for each pair $u, v \in G_k$ we have, a.a.s., that u, v are either adjacent or are the ends of a special 2-dipath. Thus a.a.s., G_k is an (m,n)-clique. Therefore almost every (m,n)-mixed graph is an (m,n)-clique.

□

3 Planar and Outerplanar (m,n)-cliques

For the family of $(1,0)$-colored mixed graphs, the largest outerplanar clique and the largest planar clique have order 7 and 15, respectively [9]. Here we refine the method used in [9] to find the exact value on the order of the largest outerplanar (n,m)-clique and bounds on the order of the largest planar (n,m)-clique.

Let the set of vertices adjacent to a vertex u by an edge of color i be denoted by $N_i(u)$ for all $i \in \{1,2,...,n\}$; the set of vertices v that are adjacent to u with an arc uv of color j be denoted by $N_j^+(u)$; and the set of vertices v that are adjacent to u with an arc vu of color j be denoted by $N_j^-(u)$ for all $j \in \{1, 2, ..., m\}$. A *dominating set* of a graph G is a set of vertices D such that every vertex of G is e ither contained in D or has a neighbor in D. A *universal vertex* v is a vertex such that $\{v\}$ is a dominating set.

Theorem 3.1. *For the family $\mathcal{O}$ of outerplanar graphs we have,*

$$\omega_{a(m,n)}(\mathcal{O}) = \omega_{r(m,n)}(\mathcal{O}) = 3(2m + n) + 1$$

for all $(m, n) \neq (0, 1)$.

Proof. We first show that $\omega_{a(m,n)}(\mathcal{O}) \geq 3(2m + n) + 1$ by giving an outerplanar (m, n)-clique on $3(2m + n) + 1$ vertices. Let Γ be the simple graph formed from $2m + n$ disjoint copies of P_3 (the path on 3 vertices), together with a universal vertex. Note that for any $(m, n) \neq (0, 1)$ a P_3 can be assigned adjacencies in such a way that it becomes a special 2-path satisfying one of the five conditions listed in the begining of Section 2.

We form H from Γ in such a way that each of $N_j^+(v)$, $N_j^-(v)$ and $N_i(v)$ induces a special 2-path for all $j \in \{1, 2, \cdots, m\}$ and $i \in \{1, 2, \cdots, n\}$ in H.

Observe that each pair of vertices are either adjacent or joined by a special 2-path. Thus,

$$\omega_{r(m,n)}(\mathcal{O}) \geq \omega_{a(m,n)}(\mathcal{O}) \geq 3(2m + n) + 1.$$

To prove the upper bound let G be a minimal (with respect to the number of vertices) (m, n)-colored mixed graph such that $\omega_{r(m,n)}(U(G)) = \omega_{r(m,n)}(\mathcal{O})$. Moreover, we may assume that $U(G)$ is maximal outerplanar. That is, no edge can be added to $U(G)$ so that it remains outerplanar. This assumption is valid; adding edges cannot increase the relative clique number of G as $\omega_{r(m,n)}(U(G)) = \omega_{r(m,n)}(\mathcal{O})$. Let $R \subseteq V$ be a relative clique of cardinality $\omega_{r(m,n)}(\mathcal{O})$ and let $S = V \setminus R$. Since $U(G)$ is maximal outerplanar, we have $d(v) \geq 2$ for all $v \in V$. Since G is outerplanar, there exists a vertex $u_1 \in V$ with $d(u_1) = 2$. Note that if $u_1 \in S$ then we can delete u_1 and connect the neighbors of u_1 with an edge (if they are not already adjacent) to obtain a graph with the same relative clique number, contradicting the minimality of G. Thus, $u_1 \in R$. Fix an outerplanar embedding of G with the outer (facial) cycle having vertices $u_1, u_2, ..., u_{|R|}$ of R embedded in a clockwise manner on the cycle. Let a and b be the neighbors of u_1. Note that a and b are adjacent and a and b can have at most one common neighbor, say x, other than u_1, as $U(G)$ is outerplanar.

Every vertex of $R \setminus \{u_1, a, b\}$ is an end of a special 2-path whose other end is u_1. Therefore, each vertex of $R \setminus \{u_1, a, b, x\}$ is adjacent to exactly one of a and b. From the first part of the proof we have

$$\omega_{r(m,n)}(\mathcal{O}) \geq \omega_{a(m,n)}(\mathcal{O}) \geq 3(2m + n) + 1.$$

Thus $|R \setminus \{u_1, a, b, x\}| \geq 3$, for all $(m, n) \neq (0, 1)$. Suppose neither a nor b is adjacent to each vertex of $R \setminus \{u_1, a, b, x\}$. In that case, there are two vertices in $R \setminus \{u_1, a, b, x\}$ that are neither adjacent nor the ends of a special 2-path. Therefore, either a or b must be adjacent to all the vertices of $R \setminus \{a, b\}$. Assume without loss of generality that a is adjacent to all the vertices of $R \setminus \{a\}$. Let u_i and u_j be a pair of distinct vertices of $R \setminus \{a\}$. If both $u_i a$ and $u_j a$ have the same adjacency type, then $|i - j| \leq 2$ when reduced modulo $|R|$, as otherwise they can be neither adjacent nor the ends of a special 2-path in G. Hence there are at most three vertices from $R \setminus \{a\}$, say r_1, r_2 and r_3, so that each of $r_1 a$, $r_2 a$ and $r_3 a$ has the same adjacency type.

From this we conclude that

$$
\begin{aligned}
|R| &\leq 2\sum_{k=1}^{m} 3 + \sum_{k=1}^{n} 3 + |\{a\}| \\
&\leq 3(2m+n)+1.
\end{aligned}
$$

□

Using this result, we prove lower and upper bounds for the (m,n)-absolute clique number of the family of planar graphs.

Theorem 3.2. *For the family $\mathcal{P}$ of planar graphs, we have*

$$3(2m+n)^2 + (2m+n) + 1 \leq \omega_{a(m,n)}(\mathcal{P}) \leq 9(2m+n)^2 + 2(2m+n) + 2$$

for all $(m,n) \neq (0,1)$.

Proof. First we show that

$$3(2m+n)^2 + (2m+n) + 1 \leq \omega_{a(m,n)}(\mathcal{P})$$

by constructing a planar (m,n)-clique $H^* = (V^*, A^* \cup E^*)$ on $3(2m+n)^2+(2m+n)+1$ vertices. Recall the outerplanar (m,n)-clique H from the proof of Theorem 3.1. We construct H^* from $2m+n$ disjoint copies of H together with a universal vertex x in such a way that each of $N_j^+(x)$, $N_j^-(x)$ and $N_i(x)$ induces the outerplanar (m,n)-clique H for all $j \in \{1,2,\cdots,m\}$ and $i \in \{1,2,\cdots,n\}$.

Observe that each pair of vertices are either adjacent or joined by a special 2-path. It is easy to check that the graph H^* is indeed planar. Therefore

$$3(2m+n)^2 + (2m+n) + 1 \leq \omega_{a(m,n)}(\mathcal{P}).$$

To prove the upper bound, first notice that the underlying simple graph of any (m,n)-clique has diameter 2. Let $G = (V, A \cup E)$ be a planar (m,n)-clique with $\omega_{a(m,n)}(G) > 3(2m+n)^2 + (2m+n)+1$. We may assume that G is maximal (i.e., triangulated), since deleting edges does not increase the (m,n)-absolute clique number. As each diameter 2 planar graph on at least 10 vertices has a dominating set of size 2 [4] and

$$\omega_{a(m,n)}(\mathcal{P}) \geq 3(2m+n)^2 + (2m+n) + 1 \geq 15,$$

we may assume that H has a dominating set of size at most 2.

First assume that G is dominated by a single vertex x. Let G' be the graph obtained by deleting x from G. Note that G' is an outerplanar graph. Furthermore, the graph induced by $N_i(x)$ is a relative (m,n)-clique of G'. Thus by Theorem 3.1, we have $|N_i(x)| \leq 3(2m+n)+1$ for all $i \in \{1,2,...,n\}$. Similarly, we have $|N_j^+(x)|, |N_j^-(x)| \leq 3(2m+n)+1$ for all $j \in \{1,2,...,m\}$. Thus

$$
\begin{aligned}
\omega_{a(m,n)}(G) &\leq |\cup_{i=1}^{n} N_i(x)| + |\cup_{j=1}^{m} N_j^+(x)| + |\cup_{j=1}^{m} N_j^-(x)| + |\{x\}| \\
&\leq 3(2m+n)^2 + (2m+n) + 1.
\end{aligned}
$$

Assume now that G has a dominating set of size 2. Let $\{x,y\} \subset V(G)$ be a dominating set that is maximum with respect to the number of common neighbors of x and y over all 2-vertex dominating sets of G. Let

- $C = N(x) \cap N(y)$ and $C_{ij} = N_i(x) \cap N_j(y)$ for all $i, j \in \{1, 2, ..., n\}$;
- $C^{*\alpha}_{ij} = N_i(x) \cap N^{\alpha}_j(y)$ and $C^{\alpha *}_{ji} = N^{\alpha}_j(x) \cap N_i(y)$ for all $i \in \{1, 2, ..., n\}$, $j \in \{1, 2, ..., m\}$ and $\alpha \in \{+, -\}$;
- $C^{\alpha\beta}_{ij} = N^{\alpha}_i(x) \cap N^{\beta}_j(y)$ for all $i, j \in \{1, 2, ..., m\}$ and $\alpha, \beta \in \{+, -\}$;
- $S_{x_i} = N_i(x) \setminus C$, $S_{y_i} = N_i(y) \setminus C$, $S^{\alpha}_{x_j} = N^{\alpha}_j(x) \setminus C$ and $S^{\alpha}_{y_j} = N^{\alpha}_j(y) \setminus C$ for all $i \in \{1, 2, ..., n\}$, $j \in \{1, 2, ..., m\}$ and $\alpha \in \{+, -\}$; and
- $S_x = N(x) \setminus C$, $S_y = N(y) \setminus C$ and $S = S_x \cup S_y$.

For $|C| \geq 6$ it must be that $|C_{ij}|, |C^{*\beta}_{ik}|, |C^{\alpha *}_{lj}|, |C^{\alpha\beta}_{lk}| \leq 3$ for all $i, j \in \{1, 2, ..., n\}$, $l, k \in \{1, 2, ..., m\}$ and $\alpha, \beta \in \{+, -\}$; it is not possible to have pairwise distance at most 2 between the vertices of C that have the same adjacency type with both x and y and also maintain the planarity of G. From this fact we conclude

$$|C| \leq \left| \bigcup_{1 \leq i,j \leq n} C_{ij} \right| + \left| \bigcup_{\substack{1 \leq i \leq n, \\ 1 \leq k \leq m, \\ \alpha \in \{+,-\}}} C^{*\beta}_{ik} \right| + \left| \bigcup_{\substack{1 \leq l \leq m, \\ 1 \leq j \leq n, \\ \alpha \in \{+,-\}}} C^{\alpha *}_{lj} \right| + \left| \bigcup_{\substack{1 \leq l,k \leq m, \\ \alpha,\beta \in \{+,-\}}} C^{\alpha\beta}_{lk} \right| \leq 3(2m+n)^2.$$

As $3(2m+n)^2 \geq 6$ for all $(m, n) \neq (0, 1)$, it follows that $|C| \leq 3(2m+n)^2$.

Consider first the case that $|C| \geq 2$. Consider a pair of vertices $u, v \in C$. Note that the cycle induced by x, y, u, v divides the plane into two regions: denote the interior by R_1 and the exterior by R_2. Consider a planar embedding of G. Observe that if we delete the vertices x and y and all the vertices placed in R_2, then the resultant graph, denoted by G_1, is outerplanar. Similarly, if we delete the vertices x, y and all the vertices placed in R_1, then the resultant graph, denoted by G_2, is outerplanar.

Observe that the set $(S_{x_i} \cup S_{y_i}) \cap V(G_1)$ induces a relative (m, n)-clique. Thus, by Theorem 3.1 we have

$$|(S_{x_i} \cup S_{y_i}) \cap V(G_1)| \leq 3(2m+n) + 1,$$

for all $i \in \{1, 2, ..., n\}$. Similarly, we can show that

$$|(S^{\alpha}_{x_j} \cup S^{\alpha}_{y_j}) \cap V(G_1)| \leq 3(2m+n) + 1,$$

for all $j \in \{1, 2, ..., m\}$ and $\alpha \in \{+, -\}$. Thus in G_1, we have $|S \cap G_1| \leq 3(2m+n)^2 + (2m+n)$. Similarly, we have $|S \cap G_2| \leq 3(2m+n)^2 + (2m+n)$. Together these two facts imply

$$|S| \leq 6(2m+n)^2 + 2(2m+n).$$

For $|C| = 1$, the graph obtained by deleting the vertices x and y is outerplanar. Repeating the argument as above yields $|S| \leq 3(2m+n)^2 + (2m+n)$. Thus, regardless of the cardinality of C, it follows that $|S| \leq 6(2m+n)^2 + 2(2m+n)$. Therefore,

$$|G| = |C| + |S| + |\{x, y\}| \leq 9(2m+n)^2 + 2(2m+n) + 2.$$

□

In the proof of Theorem 3.2 we apply the exact bounds for outerplanar graphs given in Theorem 3.1 to give a lower bound for $\omega_{a(m,n)}(\mathcal{P})$. For the cases $(m,n) = (1,0)$ and $(0,2)$ this lower bound is best possible [9, 13]. We conjecture this to be the case for all $(m,n) \neq (0,1)$.
Conjecture 3.3. *For the family $\mathcal{P}$ of planar graphs, we have*

$$\omega_{a(m,n)}(\mathcal{P}) = 3(2m+n)^2 + (2m+n) + 1$$

for all $(m,n) \neq (0,1)$.

4 Computational Complexity

It is known that the complexity of deciding whether, given an undirected simple graph G, colors may be assigned to the edges of G to make it an $(0,2)$-clique is NP-hard. A similar result holds for orienting edges to form a $(0,1)$-clique [2]. Here we address a related problem concerning signed graphs, an equivalence relation on the family of $(0,2)$-colored mixed graphs [10]. As we are able to formulate and address the problem using tools developed herein, we forgo a complete background and encourage the reader to consult [10], where the background of this class of graphs is described in full.

Let G be a 2-edge-colored graph, i.e., a $(0,2)$-colored mixed graph. An *unbalanced 4-cycle* of G is a 4-cycle of $U(G)$ having an odd number of edges of the same color in G. We call G a *signed clique* if every pair of vertices are either adjacent or belong to an unbalanced 4-cycle. In this section we show that it is NP-complete to decide whether, given an undirected graph, we can assign colors to the edges to obtain a signed clique. This implies that there should not be an easy characterization of signed cliques in terms of their underlying undirected graphs, unless $\mathsf{P} = \mathsf{NP}$.

SIGNED CLIQUE 2-EDGE-COLORING
Input: An undirected graph Γ.
Question: Does there exist a signed clique G such that $U(G) = \Gamma$?
Theorem 4.1. SIGNED CLIQUE 2-EDGE-COLORING *is NP-complete.*

Proof. It is easily seen that SIGNED CLIQUE 2-EDGE-COLORING is in NP. That this problem is NP-hard follows by reduction from the following NP-complete problem.

MONOTONE NOT-ALL-EQUAL 3-SATISFIABILITY
Instance: A 3CNF formula F over variables $x_1, x_2, ..., x_n$ and clauses $C_1, C_2, ..., C_m$ involving no negated variables.
Question: Is F *not-all-equal satisfiable*, that is, does there exist a truth assignment to the variables under which every clause has at least one true variable and at least one false variable?

As 2-COLORING OF 3-UNIFORM HYPERGRAPH is NP-complete (see [7]), is follows that MONOTONE NOT-ALL-EQUAL 3-SATISFIABILITY remains NP-complete even when restricted to formulas whose clauses have three distinct variables. As such, we may assume that within each clause of F there are three distinct variables.

From a 3CNF formula F, we construct an undirected graph G_F such that

$$F \text{ is not-all-equal satisfiable}$$
$$\Leftrightarrow$$
$$G_F \text{ can be 2-edge-colored to be a signed clique.}$$

The construction of G_F is achieved in two steps. We first construct, from F, an undirected graph H_F such that F is not-all-equal satisfiable if and only if there exists a 2-edge-coloring c_H of H_F under which only some *representative pairs* of non-adjacent vertices belong to unbalanced 4-cycles. This equivalence is obtained by designing H_F such that every representative pair belongs to a unique 4-cycle. Many of these unique 4-cycles overlap to force the color of some edges in a coloring of the edges with two colors. We then obtain G_F by adding some vertices and edges to H_F, in such a way that 1) no new 4-cycles including representative pairs are created, and 2) there exists a 2-edge-coloring of $G_F - E(H_F)$ for which every non-representative pair of vertices of G_F is contained in an unbalanced 4-cycle. In this way, the equivalence between G_F and F depends only on the equivalence between H_F and F, which is not altered when G_F is constructed from H_F.

Step 1: Constructing the core H_F

Begin with a pair of vertices r_1 and r_2. For every variable x_i of F, add a vertex u_i to H_F, as well as the edges u_ir_1 and u_ir_2. For every $i \in \{1, 2, ..., n\}$, assuming the variable x_i belongs to the (distinct) clauses $C_{j_1}, C_{j_2}, ..., C_{j_{n_i}}$, add n_i new vertices $v_{i,j_1}, v_{i,j_2}, ..., v_{i,j_{n_i}}$ to H_F and join these new vertices to both r_1 and r_2. Finally, for every clause $C_j = (x_{i_1} \vee x_{i_2} \vee x_{i_3})$ of F, add a new vertex w_j to H_F, and join it to all of $v_{i_1,j}, v_{i_2,j}, v_{i_3,j}$.

The *representative pairs* of vertices H_F are as follows. For every variable x_i of F, all pairs of the form $u_i, v_{i,j}$ are representative. Also, for every clause $C_j = (x_{i_1} \vee x_{i_2} \vee x_{i_3})$ of F, the pairs $\{v_{i_1,j}, v_{i_2,j}\}$, $\{v_{i_1,j}, v_{i_3,j}\}$ and $\{v_{i_2,j}, v_{i_3,j}\}$ are representative.

We call a 2-edge-coloring c_H *good* if each representative pair of H_F belongs to an unbalanced 4-cycle with respect to the edge colors given by c_H. Let (H_F, c_H) denote the 2-edge-colored graph obtained from H_F by coloring its edges as indicated by c_H. Given any two edges xy and $x'y$ of H_F both incident to a vertex y, we say that x and x' *agree* (resp. *disagree*) on y (with respect to c_H) if xy and $x'y$ are assigned the same color (resp. different colors) by c_H.

Claim 1. *Let c_H be a good 2-edge-coloring of H_F, and let x_i be a variable appearing in clauses $C_{j_1}, C_{j_2}, ..., C_{j_{n_i}}$ of F. If r_1 and r_2 agree (resp. disagree) on u_i, then r_1 and r_2 disagree (resp. agree) on $v_{i,j_1}, v_{i,j_2}, ..., v_{i,j_{n_i}}$.*

Proof. Recall that x_{i_1}, x_{i_2} and x_{i_3} are pairwise distinct. Assume that r_1 and r_2 agree on u_i. The claim then follows from the facts that every pair $\{u_i, v_{i,j}\}$ is representative, and the only 4-cycle of H_F including u_i and $v_{i,j}$ is $u_ir_1v_{i,j}r_2u_i$. The case that r_1 and r_2 disagree on u_i follows in a similar manner. □

Claim 2. *Let c_H be a good 2-edge-coloring of H_F, and let $C_j = (x_{i_1} \vee x_{i_2} \vee x_{i_3})$ be a clause of F. Then r_1 and r_2 cannot agree or disagree on each of $v_{i_1,j}, v_{i_2,j}, v_{i_3,j}$.*

Proof. First note that the only 4-cycles of H_F containing, say, $v_{i_1,j}$ and $v_{i_2,j}$ are $v_{i_1,j}r_1v_{i_2,j}r_2v_{i_1,j}$, $v_{i_1,j}w_jv_{i_2,j}r_1v_{i_1,j}$ and $v_{i_1,j}w_jv_{i_2,j}r_2v_{i_1,j}$. The claim then follows from the fact that if r_1 and r_2, say, agree on all of $v_{i_1,j}, v_{i_2,j}, v_{i_3,j}$, then (H_F, c_H) contains no unbalanced 4-cycle including r_1 and

r_2 and any two of $v_{i_1,j}, v_{i_2,j}, v_{i_3,j}$. So, since c_H is a good 2-edge-coloring, there is an unbalanced 4-cycle containing w_j and any two of $v_{i_1,j}, v_{i_2,j}, v_{i_3,j}$. It is easily verified that this is impossible.

Assume without loss of generality that r_1 and r_2 agree on $v_{i_1,j}$ and disagree on $v_{i_2,j}$ and $v_{i_3,j}$. Observe that $r_1 v_{i_1,j} r_2 v_{i_2,j} r_1$ and $r_1 v_{i_1,j} r_2 v_{i_3,j} r_1$ are unbalanced 4-cycles of (H_F, c_H). That c_H is good is not contradicted, since, say, $v_{i_2,j}$ and $v_{i_3,j}$ can agree on w_j (and, in such a situation, the cycle $v_{i_2,j} w_j v_{i_3,j} r_1 v_{i_2,j}$ is an unbalanced 4-cycle). Note that coloring the edges incident to w_j can only create unbalanced 4-cycles containing the representative pairs $\{v_{i_1,j}, v_{i_2,j}\}$, $\{v_{i_1,j}, v_{i_3,j}\}$ and $\{v_{i_2,j}, v_{i_3,j}\}$. So coloring the edges incident to w_j to make $v_{i_2,j}$ and $v_{i_3,j}$ belong to some unbalanced 4-cycle does not affect the existence of other unbalanced 4-cycles, including those from other representative pairs. $\square$

We claim that from a good 2-edge-coloring of H_F, we can deduce a not-all-equal truth assignment satisfying F. To see this, for every variable x_i of F, if r_1 and r_2 agree (resp. disagree) on some vertex $v_{i,j}$, then assign the value *true* (resp. *false*) to x_i in clause C_j of F. Claim 1 gives that if x_i is set to some truth value, then x_i provides the same truth value to every clause containing it. That is, the truth assignments are consistent. Claim 2 gives that every clause C_j is not-all-equal satisfied if and only if C_H is a good 2-edge-coloring. And so we produce a good 2-edge-coloring of H_F from a truth assignment not-all-equal satisfying F, and vice-versa.

Step 2: From H_F to G_F

We now construct G_F from H_F so that:

1. every 4-cycle of G_F that includes a representative pair of H_F is the only 4-cycle of H_F containing that pair, and
2. $G_F - E(H_F)$ can be 2-edge-colored, in such a way that every pair of vertices not forming a representative pair belong to some unbalanced 4-cycle.

In this way, the graph G_F will be the support of a signed clique if and only if H_F admits a good 2-edge-coloring, which is true if and only if F can be not-all-equal satisfied. The result then holds by transitivity.

To obtain G_F from H_F add vertices and edges as follows: For every vertex u of H_F, add two new vertices a_u and b_u, as well as the edges ua_u and ub_u. For every pair of vertices u, v of H_F that do not form a representative pair, add a pair of vertices $c_{u,v}$ and $c'_{u,v}$, as well as the edges $uc_{u,v}, uc'_{u,v}, vc_{u,v}, vc'_{u,v}$. Finally, add edges so that the subgraph induced by the newly added vertices (i.e., the a_u's, b_u's, $c_{u,v}$'s and $c'_{u,v}$'s) is a clique. The resulting graph is G_F. As claimed earlier, it can be checked that the only 4-cycle of G_F containing the representative pair u, v is the only 4-cycle of H_F containing u and v. This is because the shortest path from u to v in H_F has length at least 2, while the shortest path from u to v through the clique we have added has length 3.

Consider the following 2-edge-coloring of $G_F - E(H_F)$. For every vertex $u \in V(H_F)$, assign color 1 to ub_u. For every pair of distinct vertices u, v of H_F such that $\{u, v\}$ is not representative (i.e., $c_{u,v}$ exists), arbitrarily choose one of $c_{u,v}u$ and $c_{u,v}v$, and assign color 1 to that edge. Finally assign color 2 to all other edges. Clearly, under this partial 2-edge-coloring of G_F, every pair of vertices u, v of G_F not forming a representative pair are either adjacent or belong to some unbalanced 4-cycle:

- if u, v do not belong to H_F, then they belong to the clique and are hence adjacent;
- if u belongs to H_F but v does not, then observe that either u and v are adjacent (in this situation v is either a_u, b_u, $c_{u,w}$ or $c'_{u,w}$ for some w), or ua_uvb_uu is an unbalanced 4-cycle;
- if u, v are vertices of H_F and $\{u, v\}$ is not representative, then $uc_{u,v}vc'_{u,v}u$ is an unbalanced 4-cycle (in particular, that cycle has precisely only one edge assigned color 1).

By the previous arguments, finding a truth assignment not-all-equal satisfying F is equivalent to 2-edge-coloring G_F so that a signed clique is obtained. Thus SIGNED CLIQUE 2-EDGE-COLORING is NP-hard, and hence NP-complete. □

5 Comparing the Mixed Chromatic Number of $(1, 0)$-colored and $(0, 2)$-colored Mixed Graphs

Previous studies of homomorphisms of oriented graphs and 2-edge-colored graphs have shown striking similarities in both techniques and results [8, 14]. These similarities seem to stem from the two choices for adjacency type between each pair of vertices in each of these classes of graphs. In this final section we use the notions of absolute and relative clique to show that in general the parameters $\chi_{1,0}$ and $\chi_{0,2}$ may be arbitrarily far apart.

Theorem 5.1. *For every positive integer n, there exists undirected graphs G_n and H_n such that $\chi_{0,2}(G_n) - \chi_{1,0}(G_n) = 2^n$ and $\chi_{1,0}(H_n) - \chi_{0,2}(H_n) = 2^n$.*

Before providing the proof of this result, we introduce some constructions and definitions, which serve to simplify the proof of this result.

Let Γ be an undirected simple graph. The undirected simple graph $2\Gamma = \Gamma + \Gamma$ is obtained from the disjoint union of two copies, Γ_1 and Γ_2, of Γ, together with a universal vertex ∞ adjacent to every vertex of Γ_1 and Γ_2. Further, we recursively define the undirected simple graph $k\Gamma = (k-1)\Gamma + (k-1)\Gamma$. We similarly define $2G = G+G$ and $kG = (k-1)G + (k-1)G$ for a $(0, 2)$-colored mixed graph G by assigning color i $(i = 1, 2)$ to edges between ∞ and G_i.

Let G be a $(0, 2)$-colored mixed graph, with vertex set $V(G) = \{v_1, v_2, ..., v_k\}$. The $(0, 2)$-colored mixed graph G^2, with vertex set $\{v_1^1, v_2^1, ..., v_k^1\} \cup \{v_1^2, v_2^2, ..., v_k^2\}$, is formed from two copies of G, say G_1 and G_2, together with a universal vertex ∞. We add edges $v_i^1 v_j^2$ for all $v_iv_j \in E(G)$ and color these edges to agree with the color of v_iv_j in G. Finally, the edges between G_1 and ∞ are assigned color 1, and those between G_2 and ∞ are assigned color 2.

Let Γ be an undirected graph. An *(m, n)-universal chromatic bound* of Γ is an (m, n)-colored mixed graph H on $\chi_{m,n}(\Gamma)$ vertices such that $G \to H$, where G is a (m, n)-colored mixed graph and $U(X) = \Gamma$. Note that an (m, n)-universal chromatic bound may not exist for every undirected graph Γ. For instance, let $\Gamma = K_3$, we have $\chi_{0,2}(\Gamma) = 3$, but K_3 does not have a a $(0, 2)$-universal chromatic bound. That is, there is no $(0, 2)$-colored mixed graph on 3 vertices that is a homomorphic image of every $(0, 2)$-colored mixed graph that has K_3 as its underlying simple graph.

Lemma 5.2. *For $(m, n) = (0, 2)$ or $(1, 0)$, if H is an (m, n)-universal chromatic bound of Γ, then H^2 is an (m, n)-universal chromatic bound for 2Γ.*

Proof. We begin with the case $(m,n) = (0,2)$. Let H be a $(0,2)$-universal chromatic bound for Γ.

Let G' be a $(0,2)$-colored mixed graph so that $U(G') = \Gamma$ and $\chi_{0,2}(G') = \chi_{0,2}(\Gamma)$. Note that $U(2G') = 2\Gamma$. Clearly,

$$\chi_{0,2}(2\Gamma) \geq \chi_{0,2}(G') = 2\chi_{0,2}(G') + 1.$$

Let G be a $(0,2)$-colored mixed graph such that $U(G) = 2\Gamma$. There exist disjoint subgraphs, G_1 and G_2, of G such that the vertices of G_i $(i = 1,2)$ correspond to those in Γ_i in the construction of 2Γ. It suffices to show that G admits a homomorphism to H^2. As H is a $(0,2)$-universal chromatic bound for Γ, there exist homomorphisms $f_1 : G_1 \to H$ and $f_2 : G_2 \to H$. We construct a homomorphism $f : H \to H^2$ as follows:

$$f(v) = \begin{cases} f_i(v)^1 & \text{if } v \in V(Y_i) \text{ and the edge } \infty v \text{ has color 1,} \\ f_i(v)^2 & \text{if } v \in V(Y_i) \text{ and the edge } \infty v \text{ has color2,} \\ \infty & \text{if } v = \infty. \end{cases}$$

Thus, H^2 is a $(0,2)$-universal chromatic bound for 2Γ.

The proof of the case $(m,n) = (0,1)$ follows similarly by analogously defining $2G$ and G^2 for $(1,0)$-colored mixed graphs; replace edges of color 1 incident to ∞ with arcs oriented to have their head at ∞, and those of color 2 with arcs oriented to have their tail at ∞. □

Corollary 5.3. *For $(m,n) = (0,2)$ or $(1,0)$, if H is an (m,n)-universal chromatic bound of Γ, then $\chi_{m,n}(H^2) = 2\chi_{m,n}(\Gamma) + 1$.*

To proceed with the proof of Theorem 5.1 we require the following results.

Lemma 5.4 (Raspaud and Nešetřil [11])**.** *The path P_5 on 5 vertices has a $(0,2)$-universal chromatic bound on 4 vertices and a $(1,0)$-universal chromatic bound on 3 vertices.*

Note that P_5 with alternating colored edges has $(0,2)$-colored mixed chromatic number 4. Also the directed P_5 with has $(0,1)$-colored mixed chromatic number 3.

Lemma 5.5 (Bensmail [**?**], Fertin, Raspaud and Roychowdhury [**?**])**.** *The 2×4 square grid graph $G(2,4)$ has a $(0,2)$-universal chromatic bound on 5 vertices and a $(1,0)$-universal chromatic bound on 6 vertices.*

We now prove Theorem 5.1.

Proof of Theorem 5.1. All complete graphs have their $(0,2)$-colored mixed chromatic number equal to their $(1,0)$-colored mixed chromatic number. Thus we can take any complete graph as a candidate for G_0 or H_0. By taking $G_n = nP_5$ and $H_n = nG(2,4)$. The result follows from Lemmas 5.4 and 5.5 and Corollary 5.3. □

Conclusion

Given that the technique utilized both by Raspaud and Sopena [12] for oriented graphs and by Alon and Marshall [1]for k-edge-colored graphs was generalized by Nešetřil and Raspaud [11] for (m,n)-colored mixed graphs, it seems likely that the similarly in techniques and methods in the study of oriented graphs and 2-edge-colored graphs comes from the more general structure of (m, n)-colored mixed graphs. That the definitions of graph, homomorphism, chromatic number, and now clique, may be generalized to capture these notions for simple graphs, oriented graphs and k-edge-colored graphs, suggests that further study in to (m, n)-colored mixed graphs will provide insight into open problems for each of these types of graphs.

References

[1] N. Alon and T. H. Marshall. Homomorphisms of edge-colored graphs and Coxeter groups. *Journal of Algebraic Combinatorics*, 8(1):5–13, 1998.

[2] J. Bensmail, R. Duvignau, and S. Kirgizov. The complexity of deciding whether a graph admits an orientation with fixed weak diameter. Preprint available at http://hal.archives-ouvertes.fr/hal-00824250, 2013.

[3] O. V. Borodin. On acyclic colorings of planar graphs. *Discrete Mathematics*, 25(3):211–236, 1979.

[4] W. Goddard and M. A. Henning. Domination in planar graphs with small diameter. *Journal of Graph Theory*, 40:1–25, May 2002.

[5] W. F. Klostermeyer and G. MacGillivray. Analogs of cliques for oriented coloring. *Discussiones Mathematicae Graph Theory*, 24(3):373–388, 2004.

[6] W. F. Klostermeyer and G. MacGillivray. Homomorphisms and oriented colorings of equivalence classes of oriented graphs. *Discrete Mathematics*, 274(1–3):161–172, 2004.

[7] L. Lovász. Coverings and coloring of hypergraphs. *Proceedings of the Fourth South-eastern Conference on Combinatorics, Graph Theory, and Computing, Boca Raton, Florida*, page 312, 1973.

[8] A. Montejano, P. Ochem, A. Pinlou, A. Raspaud, and É. Sopena. Homomorphisms of 2-edge-colored graphs. *Discrete Applied Mathematics*, 158(12):1365–1379, 2010.

[9] A. Nandy, S. Sen, and É. Sopena. Outerplanar and planar oriented cliques (accepted). *Journal of Graph Theory*, 2015.

[10] R. Naserasr, E. Rollová, and É. Sopena. Homomorphisms of signed graphs. (preprint). 2013.

[11] J. Nešetřil and A. Raspaud. Colored homomorphisms of colored mixed graphs. *Journal of Combinatorial Theory, Series B*, 80(1):147–155, 2000.

[12] A. Raspaud and É. Sopena. Good and semi-strong colorings of oriented planar graphs. *Information Processing Letters*, 51(4):171–174, 1994.

[13] S. Sen. *A contribution to the theory of graph homomorphisms and colorings.* PhD thesis, University of Bordeaux, France, 2014.

[14] E. Sopena. Homomorphisms and colourings of oriented graphs: An updated survey. *Discrete Mathematics*, (0):–, 2015.